Knot colourings, self-distributivity and associated groups of quandles

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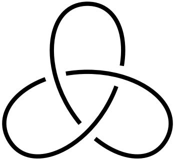
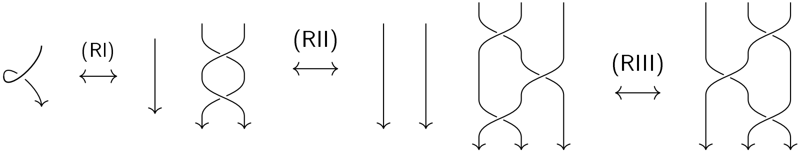
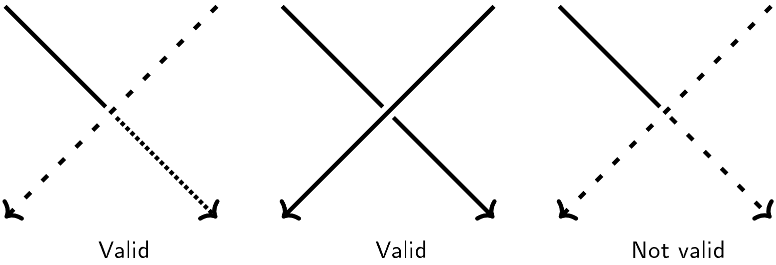
Reidemeister moves





Knot colourings



Question : Is the trefoil knot trivial ?

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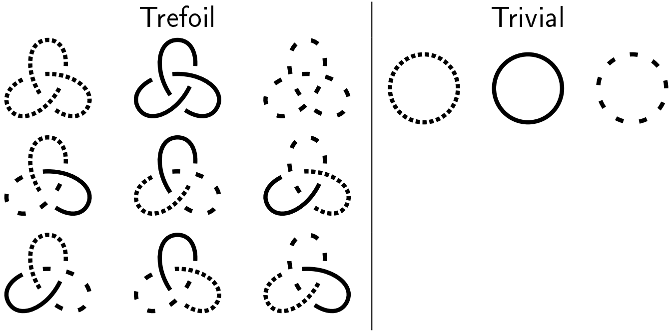
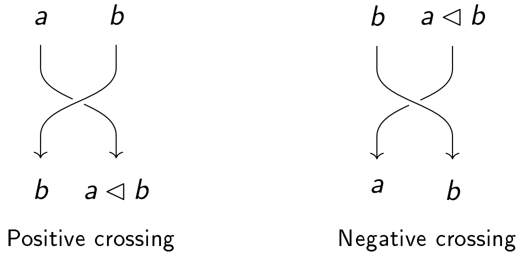


Colourings

X a set, $a, b \in X$



Invariant : number of colourings of a diagram by X

Quandles

A *quandle* is a set X with a law $\lhd$ which satisfies the following properties :

1. $a \lhd a = a$
2. $\phi_b : \begin{array}{ccc} X & \longrightarrow & X \\ a & \longmapsto & a \lhd b \end{array}$ is bijective.
3. $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$

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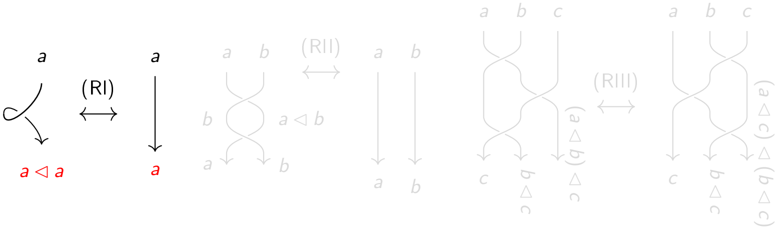

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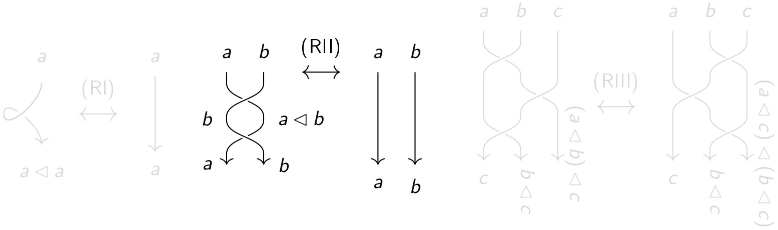

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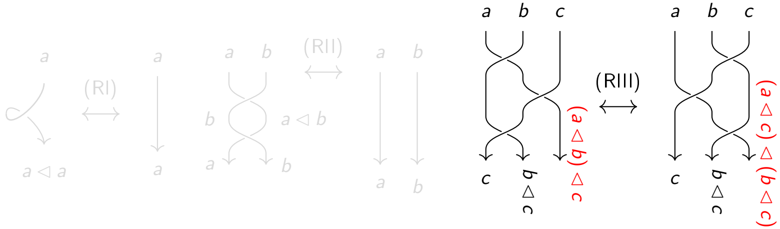

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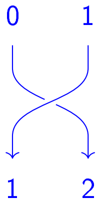
Note D_3 the dihedral quandle (Alexander quandle with $t = -1$) of order 3.



Structure group and inner automorphisms group

$$G_X = \langle e_b, b \in X \mid e_a e_b = e_b e_{a \lhd b} \rangle \qquad B_3$$

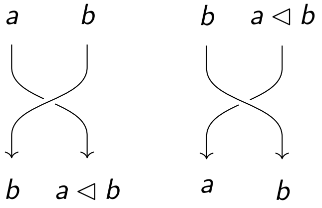
$$\downarrow \qquad\qquad \downarrow$$

$$\mathrm{Inn}(X) = \left\langle \phi_b : \begin{array}{ccc} X & \longrightarrow & X \\ a & \longmapsto & a \lhd b \end{array} \right\rangle \subset S_X \qquad S_3$$

- G_X and $\mathrm{Inn}(X)$ act on X from the right
- These actions give the same orbits

Second cohomology group

$\phi : X \times X \longrightarrow A$



Contributes for $+\phi(a, b)$.

Contributes for $-\phi(a, b)$.

2-cocycle condition :

$$\phi(a, b) + \phi(a \triangleleft b, c) = \phi(a, c) + \phi(a \triangleleft c, b \triangleleft c).$$

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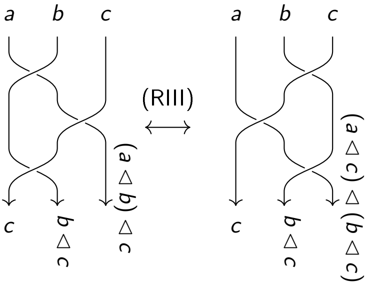

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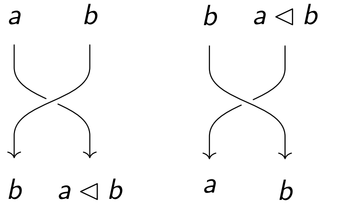

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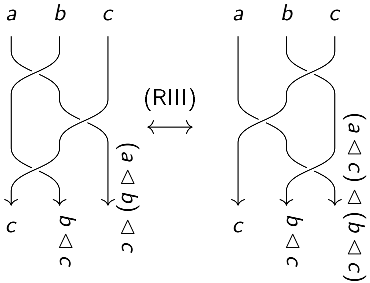

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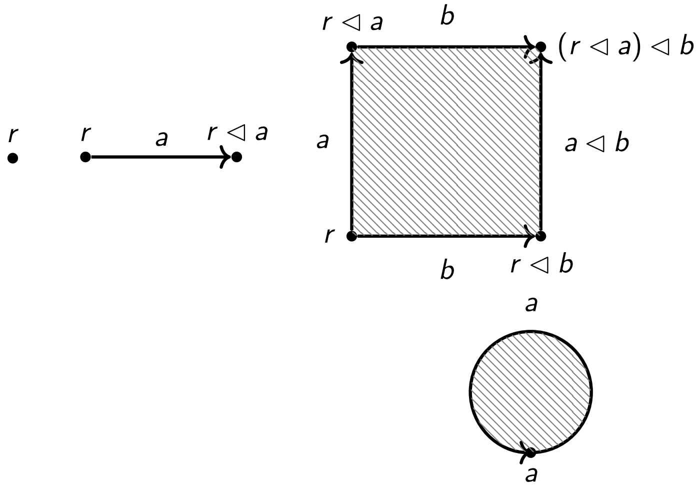
Theorem (Eisermann, 06)

Let $(X, \triangleleft)$ be a quandle. Note $\mathcal{O}$ the set of the orbits of X under the action of G_X, and $(x_\alpha)_{\alpha\in\mathcal{O}}$ be a family of representatives of the orbits. Then

$$H_2^Q(X) \cong \bigoplus_{\alpha\in\mathcal{O}} [\mathrm{Stab}_{G_X}(x_\alpha) \cap \ker(\varepsilon)]_{Ab}$$

Sketch of the proof



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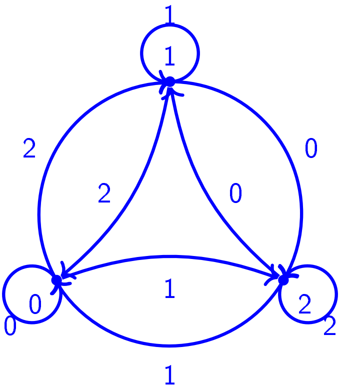
D_3



$$H_2^Q(X) \cong \bigoplus_{\alpha \in \mathcal{O}} \left[\pi_1(\mathrm{Cayl}^Q(X), x_\alpha)\right]_{Ab} \cong \bigoplus_{\alpha \in \mathcal{O}} [\mathrm{Stab}_{G_X}(x_\alpha) \cap \ker(\varepsilon)]_{Ab}$$

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Theorem

If n is prime and $t = -1$, then $\forall g \in G_X$,

$$g = e_0^{\varepsilon(g)-1} e_d$$

where d is the preserved quantity. Then, we have

$$H_2^Q(X) \cong \{0\}.$$

Rewriting techniques for n prime and $t = 1 - t$

Theorem (Clauwens, 2010)

Let $Al_{n,t}$ be the Alexander quandle of cardinal n prime and parameter t. Then

$$G_{Al_{n,t}} \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}$$

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The group presentation of G_X gives an action of B_n on X^n by $\sigma_i \cdot (x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i+1}, x_i \lhd x_{i+1}, \ldots, x_n)$.

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Then,

$$e_i e_{-ti} = e_0^2.$$

Thank you for your attention !